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Note on eigenvalue bounds for the Orr–Sommerfeld equation

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Bounds for the complex wave velocity c, determined by the Orr–Sommerfeld equation and the boundary conditions for channel flow, have been given by Joseph (1968*a*, *b*). In these notes it is shown how two of Joseph's theorems can be uniformly improved.

1. Preliminary

The differential system considered consists of the Orr–Sommerfeld equation

$$i\alpha R[(U-c)(\phi''-\alpha^2\phi)-U''\phi] = \phi^{\text{iv}}-2\alpha^2\phi''+\alpha^4\phi \tag{1}$$

and the boundary conditions

$$\phi(\pm\tfrac{1}{2}) = 0 = \phi'(\pm\tfrac{1}{2}), \tag{2}$$

for flow between parallel plates. In (1) α is the wave-number, U the dimensionless velocity of the primary flow, R the Reynolds number based on the spacing d of the plates, $c = c_r + ic_i$ is the complex wave velocity, and accents indicate differentiation with respect to the dimensionless ordinate y measured in the direction normal to the plates. For convenience, the space occupied by the fluid is specified by the interval

$$-\tfrac{1}{2} \leqslant y \leqslant \tfrac{1}{2}, \tag{3}$$

instead of $0 \leqslant y \leqslant 1$, as in the paper of Joseph (1968). The length scale remains the same. The parameters R and α are non-negative.

By multiplying (1) by ϕ^*, the complex conjugate of ϕ, and integrating throughout the interval specified by (3), using (2) whenever necessary, Synge (1938) obtained

$$c_i = \{Q-Q^*-(\alpha R)^{-1}(I_2^2+2\alpha^2I_1^2+\alpha^4I_0^2)\}/(I_1^2+\alpha^2I_0^2), \tag{4}$$

and

$$c_r = \{\textstyle\int[U|\phi'|^2+(\alpha^2U+\tfrac{1}{2}U'')|\phi|^2]\,dy\}/(I_1^2+\alpha^2I_0^2), \tag{5}$$

in which

$$I_2^2 = \int|\phi''|^2dy, \quad I_1^2 = \int|\phi'|^2dy,$$
$$I_0^2 = \int|\phi|^2dy, \quad Q = \tfrac{1}{2}i\textstyle\int U'\phi\phi'^*dy.$$

The upper limit in all the integrals is $\frac{1}{2}$ and the lower limit $-\frac{1}{2}$.

Using (4) and Schwarz's inequality, Synge (1938) obtained the estimate

$$c_i \leqslant \frac{qI_0I_1-(\alpha R)^{-1}(I_2^2+2\alpha^2I_1^2+\alpha^4I_0^2)}{I_1^2+\alpha^2I_0^2}, \tag{6}$$

where

$$q = \max|U'(y)|$$

in the interval (3).

2. An upper bound for c_i

Using (6), Joseph (1969) concluded that

$$c_i \leqslant \frac{q}{2\alpha} - \left\{\frac{\pi^2(4\pi^2+\alpha^2)}{\pi^2+\alpha^2} + \alpha^2\right\} \Big/ \alpha R, \tag{7}$$

and that, if

$$\alpha R q < f(\alpha) \equiv \max\,[M_1, M_2], \tag{8}$$

$$\left.\begin{aligned} M_1 &= (4{\cdot}73)^2\, 2\pi + 2^{\frac{3}{2}}\alpha^3, \\ M_2 &= (4{\cdot}73)^2\, 2\pi + 2\alpha^2\pi, \end{aligned}\right\} \tag{9}$$

then c_i cannot be positive. Result (8) greatly improves the result of Synge (1938).

We shall show that (7) to (9) can be uniformly sharpened. Starting from (6), we immediately obtain

Theorem 1.

$$c_i \leqslant \frac{q}{2\alpha} - \frac{\lambda^2}{\alpha R}, \tag{10}$$

in which

$$\lambda^2 = \min \frac{I_2^2 + 2\alpha^2 I_1^2 + \alpha^4 I_0^2}{I_1^2 + \alpha^2 I_0^2}. \tag{11}$$

Of course λ^2 must be given explicitly in terms of α. To evaluate λ^2, we shall use the variational method. That is, we shall give ϕ a variation $\delta\phi$ satisfying

$$\delta\phi(\pm\tfrac{1}{2}) = \delta\phi'(\pm\tfrac{1}{2}) = 0, \tag{12}$$

and require the ratio in (11) to be a minimum, thereby finding a differential equation to be satisfied by ϕ and containing λ^2 as a parameter. This equation is no longer (1). It, with (2), will determine λ^2. Since the ϕ in (1) is four-times differentiable, we shall assume $\delta\phi$ to be four times differentiable also. Remembering the definitions of I_0, I_1 and I_2, allowing ϕ to have the variation $\delta\phi$ satisfying (12) and four times differentiable but otherwise arbitrary, and requiring the ratio in (11) to be an extremum, we obtain, upon neglect of quadratic terms in $\delta\phi$ and its derivatives and after integrations by parts whenever necessary,

$$\frac{2}{I_1^2 + \alpha^2 I_0^2} \int (D^2 - \alpha^2 + \lambda^2)\,(D^2 - \alpha^2)\,\phi\,\delta\phi\,dy = 0,$$

the limits of integration being understood, and D denoting d/dy. Since $\delta\phi$ is arbitrary, ϕ must satisfy

$$(D^2 - \alpha^2 + \lambda^2)\,(D^2 - \alpha^2)\,\phi = 0. \tag{13}$$

This and (2) constitute a differential system which defines an eigenvalue problem, with λ^2 as the eigenvalue for any given α^2. The differential system admits even or odd solutions for ϕ. For even ϕ, it gives the secular equation

$$\surd(\lambda^2 - \alpha^2) \tan \tfrac{1}{2}\surd(\lambda^2 - \alpha^2) = -\alpha \tanh \tfrac{1}{2}\alpha. \tag{14}$$

The solution of (14) will be denoted by λ_e, the subscript meaning 'even'. The lowest λ_e^2 is plotted in figure 1 for comparison with the corresponding values

$$\pi^2 + \alpha^2 \quad \text{and} \quad \pi^2(4\pi^2 + \alpha^2)/(\pi^2 + \alpha^2) + \alpha^2,$$

given by Joseph in (1968) and in equation (7), respectively. That λ_e^2 is uniformly an improvement of (7) is evident.

For odd ϕ the secular equation is

$$\sqrt{(\lambda^2-\alpha^2)}\tanh\tfrac{1}{2}\alpha = \alpha\tan\tfrac{1}{2}\sqrt{(\lambda^2-\alpha^2)}. \quad (15)$$

The solution of this equation will be denoted by λ_o^2, the subscript meaning 'odd'. The values of the lowest λ_o^2 for various values of α^2 are also plotted in figure 1. It can be seen from figure 1 that the lowest λ_o^2 is greater than the lowest λ_e^2 for all values of α^2. It is also clear that the λ^2 in (13) is an extremum only if ϕ is even

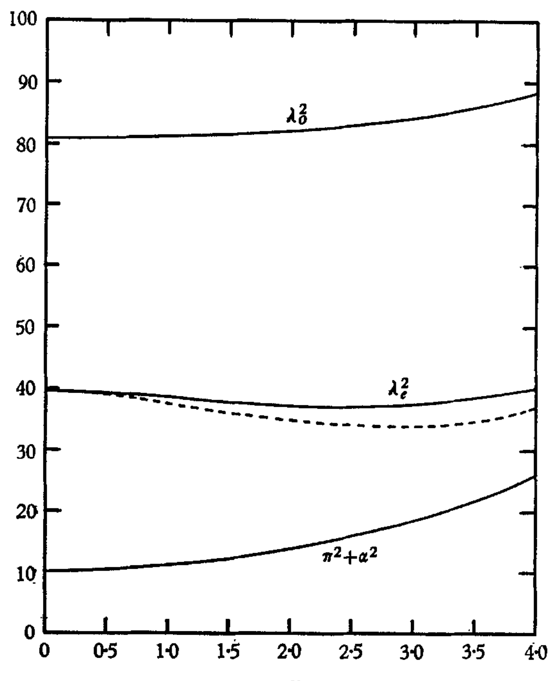

FIGURE 1. The values of Joseph's bound $\alpha^2+\pi^2(4\pi^2+\alpha^2)/(\pi^2+\alpha^2)$: - - - -. Joseph previously gave the less sharp bound $\pi^2+\alpha^2$. The improved bound is λ_e^2.

or odd, since, as can be shown, the general secular equation in the form of a four-by-four determinant can be factorized into two equations which are precisely (14) and (15). Near any solution of (14) λ^2 cannot be an extremum unless ϕ is even, and near any solution of (15) λ^2 cannot be an extremum unless ϕ is odd. In fact, the spectrum of λ_e^2 and the spectrum of λ_o^2 separate each other. Hence the lowest λ_e^2 is the value we want.

Note that from (13) and its boundary conditions we can easily obtain (with limits and dy omitted)

$$\int|D^2\phi|^2+(2\alpha^2-\lambda^2)\int|D\phi|^2+\alpha^2(\alpha^2-\lambda^2)\int|\phi|^2 = 0,$$

from which it is obvious that λ^2 is real. Thus it is quite unnecessary to consider complex forms of the function ϕ, for its real and imaginary parts would separately satisfy (13) and its boundary conditions, and the function that gives the lowest λ^2 is proportional to the real eigenfunction ϕ corresponding to the lowest eigenvalue λ_e^2. The constant of proportionality may be complex, but the lowest λ^2 is just the λ_e^2 we have obtained.

3. A sufficient condition for stability

We shall now give the improvement of (8) and (9). From (6), we see that c_i cannot be positive if

$$\alpha Rq \leqslant \frac{I_2^2+2\alpha^2I_1^2+\alpha^4I_0^2}{I_0I_1}. \tag{16}$$

We shall try to minimize the right-hand side of (16), the minimum value of which will be denoted by κ^2. If

$$\kappa_1^2 = \min\frac{I_2^2+2\alpha^2I_1^2+\alpha^4I_0^2}{I_0^2} \tag{17}$$

and

$$\kappa_2^2 = \min\frac{I_2^2+2\alpha^2I_1^2+\alpha^4I_0^2}{I_1^2}, \tag{18}$$

then obviously

$$\kappa_1\kappa_2 \leqslant \kappa^2. \tag{19}$$

The obviously correct statement, that c_i cannot be positive if

$$\alpha Rq \leqslant \kappa^2, \tag{20}$$

can be replaced by the less sharp

THEOREM 2*a*. c_1 *cannot be positive if*

$$\alpha Rq \leqslant \kappa_1\kappa_2 \tag{21}$$

(less sharp, because of (19)).

The estimate (21), however, has the advantage that κ_1 and κ_2 can be simply evaluated. The method of determining κ_1^2 and κ_2^2 is the same as that used to determine λ^2 in the preceding section. Again only even functions ϕ need be considered. The differential system determining κ_1^2 is

$$(D^2-\alpha^2)^2\phi-\kappa_1^2\phi = 0, \tag{22}$$

in conjunction with (2), and the differential system determining κ_2 is

$$(D^2-\alpha^2)^2\phi+\kappa_2^2D^2\phi = 0, \tag{23}$$

in conjunction with (2). The product $\kappa_1\kappa_2$ is plotted against α in figure 2, which also shows M_1 and M_2 given by (9). That the present estimate is an improvement over Joseph's (1969) is evident for $\alpha \leqslant 2{\cdot}4$; but for $\alpha > 2{\cdot}4$ Joseph's M_1 is a better bound. We shall now proceed to find a bound for αRq for stability which is uniformly better than Joseph's.

Since for any real b

$$I_0I_1 \leqslant \frac{1}{2b}(I_1^2+b^2I_0^2),$$

if we define $K(\alpha,b)$ by

$$K(\alpha,b) = \min\frac{2b(I_2^2+2\alpha^2I_1^2+\alpha^4I_0^2)}{I_1^2+b^2I_0^2},$$

for any real α and b, it is evident that $\kappa^2 \geqslant K(\alpha, b)$ for *all* values of b. Hence

$$\kappa^2 \geqslant K_{\max},$$

where $K_{\max}$ is the maximum of K with respect to b, for any α^2, and we can use $K_{\max}$ as a safe and at the same time good substitute for κ^2. Using the variational method, we obtain, for the determination of $K(\alpha, b)$, the differential system

$$\phi^{\text{iv}} - 2\alpha^2\phi'' + \alpha^4\phi + \frac{K}{2b}(\phi'' - b^2\phi) = 0,$$

$$\phi(\pm\tfrac{1}{2}) = 0 = \phi'(\pm\tfrac{1}{2}).$$

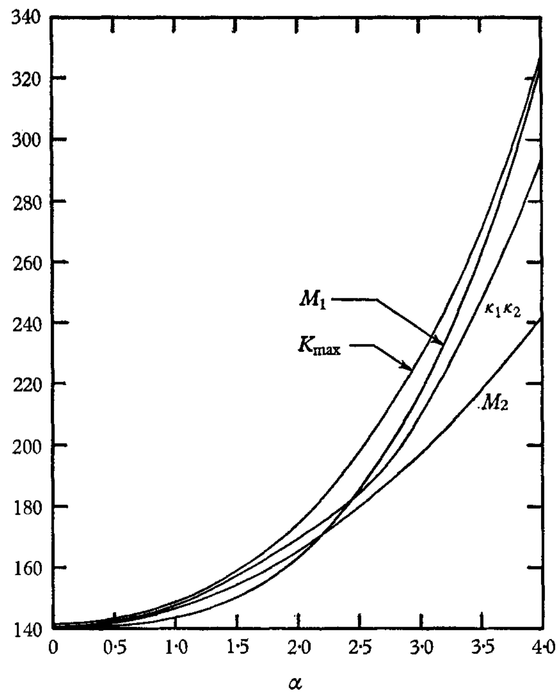

FIGURE 2. The greater of M_1 and M_2 is Joseph's bound. The improved bound is $K_{\max}$.

We shall again consider ϕ to be even, for an odd ϕ will give higher eigenvalues for K. In this way we can find $K(\alpha, b)$ for given α and various values of b. Thus $K_{\max}$ is obtained, which is a function of α only. Its values are plotted in figure 2. All of these values correspond to the value of 3·55 for b, which does not seem to vary with α in the range of calculation. It is evident that $K_{\max}$ improves Joseph's estimate (9) uniformly. We have now the sharper

THEOREM 2. *c_i cannot be positive if*

$$\alpha Rq \leqslant K_{\max}. \tag{24}$$

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computational assistance. To avoid duplicating the graphs, the numerical values of λ_e^2, λ_o^2, κ_1, κ_2, $\kappa_1\kappa_2$ and $K_{\max}$ have not been reproduced here in tabular form. Readers interested in these values are invited to write to the author.

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